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FORMAL LANGUAGES, PUSHDOWN-AUTOMATA AND C^*-ALGEBRAS

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0. INTRODUCTION

This manuscript is a survey of a talk in the RIMS workshop on Development of Operator Algebras, Sep.7-9, 2005. A part of the results written here is based on a joint research with Wolfgang Krieger. Some of details will be written in [Ma5].

N. Chomsky has classified formal languages into the following four classes:

(1) Regular languages
(2) Context free languages
(3) Contex sensitive languages
(4) Phase structure languages

such that

$$(1) \subset (2) \subset (3) \subset (4)$$

by the grammars that generate the languages (cf. [HU]). Each of the classes has a machine (algorithm) by which the languages are acceptable. The machines are

(1) Finite automata
(2) Pushdown automata
(3) Linear bounded Turing machines
(4) Turing machines

respectively. This means that regular languages are acceptable by finite automata exactly, context free languages are acceptable by pushdown automata, and so on. A finite automaton is a finite labeled graph with a distinguished initial state and a distinguished subset of terminal vertices. If we consider the situation that all vertices are both initial states and terminal states, the language accepted by such automaton is the set of admissible words of the sofic shift presented by the labeled graph. Conversely the admissible words of a sofic shift is realized as the language accepted by such finite automata. W. Krieger was the first to observe this connection between sofic shifts and regular languages ([Kr2]).

Sofic shifts are realized as finite labeled graphs, and the finite labeled graphs yield Cuntz-Krieger algebras (cf. [Iz], [Ca],[Ma5],Tom]). The author in [Ma] has generalized finite labeled graphs to λ-graph systems, and constructed C^*-algebras from

λ-graph systems ([Ma2]). We will construct a λ-graph system from a pusudown-automaton, so that pushdown automata yield C^*-algebras (cf. [KM2]).

For an $N \times N$ irreducible matrix A with entries in $\{0,1\}$, a pushdown automata M_A is constructed such that its presenting language is the language generated by the generators $S_1^*, \dots, S_N^*, S_1, \dots, S_N$ of the Cuntz-Krieger algebra $\mathcal{O}_A$. The associated C^*-algebra is simple purely infinite and does not differ from Cuntz-Krieger algebras, and the associated subshift D_A is a topological Markov shift version of the Dyck shifts D_N.

1. Languages

Let Σ be a finite set of symbols. The set Σ is called an alphabet. For $l \in \mathbb{N}$, the set $\Sigma^l = \{\mu_1 \cdots \mu_l \mid \mu_i \in \Sigma\}$ is called words of length l. We put $\Sigma^0 = \{\epsilon\}$ called the empty word. Let Σ^* be the Kleenean clousure $\cup_{l=0}^{\infty}\Sigma^l$ of Σ. A *formal language* of Σ is defined to be a subset L of Σ^*. Put for $l \in \mathbb{Z}_+$, $B_l(L) = L \cap \Sigma^l$ the set of all admissible words of length l. A formal language L over Σ is said to be *prolongable* if L is not empty and for any $w \in L$ there exist $w', w'' \in \cup_{l=1}^{\infty}\Sigma^l$ such that $w'ww'' \in L$. Namely any word of L can be extended in both right and left as an admissible word of L. Put for a nonempty formal language L over Σ

$$S(L) = \{x \in \Sigma^* \mid \text{there exists a word } w \text{ of } L \text{ such that } x \text{ is a subword of } w\}$$

Put $\mathfrak{F}(L) = S(L)^c$ in Σ^*. Define Λ_L to be the subshift whose forbidden words are $\mathfrak{F}(L)$. Denote by Λ_L^* the set of all admissible words of Λ_L. That is the set of words of Σ which are not forbidden. Hence $\Lambda_L^* = S(L)$.

Proposition 1. *Λ_L defines a non empty subshift such that $\Lambda_L^* \supset L$ if and only if L is prolongable.*

In what follows that L is prolongable formal language over Σ. By the preceding proposition, L defines a symbolic dynamics Λ_L. The symbolic dynamics Λ_L is called a symbolic dynamics generated by a formal language L. Let L_1, L_2 be prolongable formal languages over Σ_1, Σ_2 respectively. We say that L_1 is isomorphic to L_2 if there exists a bijection Φ from Σ_1 to Σ_2 that defines a symbolic conjugacy $\Phi_\infty : \Lambda_{L_1} \to \Lambda_{L_2}$ such that $\Phi_\infty((a_n)_{n\in\mathbb{Z}}) = (\Phi(a_n)_{n\in\mathbb{Z}})$ between the associated subshifts Λ_{L_1} and Λ_{L_2}. In this case, we write $L_1 \cong L_2$.

2. λ-graph systems and Dyck shifts D_N

A downward λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over an alphabet Σ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling map $\lambda : E \to \Sigma$ and a surjective map $\iota_{l,l+1} : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$. The sets V_l and $E_{l,l+1}$ are finite for each $l \in \mathbb{Z}_+$. An edge $e \in E_{l,l+1}$ has its source vertex $s(e)$ in V_l, its terminal vertex $t(e)$ in V_{l+1} and its label $\lambda(e)$ in Σ. The edges with its labeling and the map ι must satisfy a certain compatiblity condition called local property (see [Ma]). The λ-graph systems considered in [Ma] are downward λ-graph systems. Contrary an upward λ-graph systems are similarly defined such as an edge $e \in E_{l+1,l}$ has its source vertex $s(e)$ in V_{l+1} and its terminal vertex $t(e)$ in V_l. The λ-graph systems considered in [KM] are upward λ-graph systems. In what follows, we mean by a λ-graph system a downward λ-graph sytem unless we specify. A λ-graph system yields a subshift by taking the set of all label sequences appearing in the labeled Bratteli diagram.

Let us consider the Dyck shift D_N for $N \geq 2$ with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_1, \cdots, \alpha_N\}, \Sigma^+ = \{\beta_1, \cdots, \beta_N\}$. The symbols α_i, β_i correspond to the brackets $(_i,)_i$ respectively. The Dyck inverse monoid ([Kr3],[Kr4]) has the relations

$$\alpha_i \beta_j = \begin{cases} \mathbf{1} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

for $i, j = 1, \ldots, N$ and a word $\gamma_1 \cdots \gamma_n$ of Σ is admissible for D_N precisely if $\prod_{m=1}^n \gamma_m \neq 0$. For a word $\omega = \omega_1 \cdots \omega_n$ of Σ, we denote by $\tilde{\omega}$ its reduced form. Namely $\tilde{\omega}$ is a word of $\Sigma \cup \{0, \mathbf{1}\}$ obtained after the operations (2.1). Hence a word ω of Σ is forbidden for D_N if and only if $\tilde{\omega} = 0$.

Let us describe the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ of D_N introduced in [KM]. Let Σ_N be the full N-shift $\{1, \ldots, N\}^{\mathbb{Z}}$. We denote by $B_l(D_N)$ and $B_l(\Sigma_N)$ the set of admissible words of length l of D_N and that of Σ_N respectively. The vertices V_l of $\mathfrak{L}^{Ch(D_N)}$ at level l are given by the words of length l consisting of the symbols of Σ^+. That is,

$$V_l = \{(\beta_{\mu_1} \cdots \beta_{\mu_l}) \in B_l(D_N) \mid \mu_1 \cdots \mu_l \in B_l(\Sigma_N)\}.$$

Hence the cardinal number of V_l is N^l. The mapping $\iota (= \iota_{l,l+1}) : V_{l+1} \to V_l$ deletes the rightmost symbol of a word such as

$$\iota((\beta_{\mu_1} \cdots \beta_{\mu_{l+1}})) = (\beta_{\mu_1} \cdots \beta_{\mu_l}) \quad \text{for} \quad (\beta_{\mu_1} \cdots \beta_{\mu_{l+1}}) \in V_{l+1}.$$

There exists an edge labeled α_j from $(\beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_l$ to $(\beta_{\mu_0} \beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_{l+1}$ precisely if $\mu_0 = j$, and there exists an edge labeled β_j from $(\beta_j \beta_{\mu_1} \cdots \beta_{\mu_{l-1}}) \in V_l$ to $(\beta_{\mu_1} \cdots \beta_{\mu_{l+1}}) \in V_{l+1}$. It is easy to see that the resulting labeled Bratteli diagram with ι-map becomes a λ-graph system over Σ, denoted by $\mathfrak{L}^{Ch(D_N)}$, that presents the Dyck shift D_N ([KM]).

Let $A = [A(i,j)]_{i,j=1,\ldots,N}$ be an $N \times N$ matrix with entries in $\{0, 1\}$. Consider the Cuntz-Krieger algebra $\mathcal{O}_A$ for the matrix A that is the universal C^*-algebra generated by N partial isometries $t_1, \ldots, t_N$ subject to the following relations:

$$\sum_{j=1}^N t_j t_j^* = 1, \qquad t_i^* t_i = \sum_{j=1}^N A(i,j) t_j t_j^* \quad \text{for } i = 1, \ldots, N \tag{2.2}$$

([CK]). Define a correspondence $\varphi_A : \Sigma \longrightarrow \{t_i^*, t_i \mid i = 1, \ldots, N\}$ by setting

$$\varphi_A(\alpha_i) = t_i^*, \qquad \varphi_A(\beta_i) = t_i \quad \text{for } i = 1, \ldots, N.$$

Define the set

$$\mathfrak{F}_A = \{\gamma_1 \cdots \gamma_n \in \Sigma^* \mid \varphi_A(\gamma_1) \cdots \varphi_A(\gamma_n) = 0\}.$$

Let D_A be the subshift over Σ whose fobidden words are $\mathfrak{F}_A$. The subshift is called the topological Markov Dyck shift defined by A, or the vertex Dyck shift defined by A. These kinds of subshifts have first appeared in [HIK] in semigroup setting and in [KM2] in more general setting without using C^*-algebras. If all entries of A are 1, the subshift D_A becomes the Dyck shift D_N with $2N$ bracket, because the partial isometries $\{\varphi_A(\alpha_i), \varphi(\beta_i) \mid i = 1, \dots, N\}$ yield the Dyck inverse monoid. We note the fact that $\alpha_i\beta_j \in \mathfrak{F}_A$ if $i \neq j$, and $\alpha_{i_n} \cdots \alpha_{i_1} \in \mathfrak{F}_A$ if and only if $\beta_{i_1} \cdots \beta_{i_n} \in \mathfrak{F}_A$. Consider the following two subsystems of D_A

$$\Lambda_A^D = \{(\gamma_i)_{i\in\mathbb{Z}} \in D_A \mid \gamma_i \in \Sigma^+, i \in \mathbb{Z}\},$$
$$\Lambda_{A^t}^D = \{(\gamma_i)_{i\in\mathbb{Z}} \in D_A \mid \gamma_i \in \Sigma^-, i \in \mathbb{Z}\}.$$

The subshift Λ_A^D is identified with the topological Markov shift

$$\Lambda_A = \{(x_i)_{i\in\mathbb{Z}} \in \{1, \dots, N\}^{\mathbb{Z}} \mid A(x_i, x_{i+1}) = 1, i \in \mathbb{Z}\}$$

defined by the matrix A and similarly $\Lambda_{A^t}^D$ is identified with the topological Markov shift Λ_{A^t} defined by the transposed matrix A^t of A. Hence the subshift D_A contains the both topological Markov shifts Λ_A and Λ_{A^t} that do not intersect each other.

Proposition 2. *If A satisfies condition (I) in the sense of Cuntz-Krieger [CK], the subshift D_A is not sofic.*

We will define the Cantor horizon λ-graph systems $\mathfrak{L}^{Ch(D_A)}$ for the topological Markov Dyck shifts D_A. We denote by $B_l(D_A)$ and $B_l(\Lambda_A)$ the set of admissible words of length l of D_A and that of Λ_A respectively. The vertices V_l of $\mathfrak{L}^{Ch(D_A)}$ at level l are given by the admissible words of length l of Λ_A. That is,

$$V_l = \{(\beta_{\mu_1} \cdots \beta_{\mu_l}) \in B_l(D_N) \mid \mu_1 \cdots \mu_l \in B_l(\Lambda_A)\}.$$

The mapping $\iota (= \iota_{l,l+1}) : V_{l+1} \to V_l$ deletes the rightmost symbol of a word such as

$$\iota((\beta_{\mu_1} \cdots \beta_{\mu_{l+1}})) = (\beta_{\mu_1} \cdots \beta_{\mu_l}) \quad \text{for} \quad (\beta_{\mu_1} \cdots \beta_{\mu_{l+1}}) \in V_{l+1}.$$

There exists an edge labeled α_j from $(\beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_l$ to $(\beta_{\mu_0}\beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_{l+1}$ precisely if $\mu_0 = j$, and there exists an edge labeled β_j from $(\beta_j\beta_{\mu_1} \cdots \beta_{\mu_{l-1}}) \in V_l$ to $(\beta_{\mu_1} \cdots \beta_{\mu_{l+1}}) \in V_{l+1}$ precisely if $j\mu_1 \cdots \mu_{l+1} \in B_{l+2}(\Lambda_A)$. It is easy to see that the resulting labeled Bratteli diagram with ι-map becomes a λ-graph system over Σ.

Proposition 3. *The λ-graph system $\mathfrak{L}^{Ch(D_A)}$ presents the subshift D_A.*

3. Pushdown-automata

A deterministic pushdown-automaton $M = (Q, \Gamma, \Sigma, \delta)$ means that Q is a finite set of states, Γ is a finite set of stack symbols, Σ is a finite set of alphabet and δ is a finite set of transition rule such that for $a \in \Sigma$ there exists a subset $D_a \subset Q \times \Gamma$ such that

$$\delta_a : D_a \to Q \times \Gamma^*$$

where $\Gamma^* = \cup_{k=0}^{\infty}\Gamma^k$ the set of all finite words of Γ with the empty word $\Gamma^0 = \{\emptyset\}$. For $a \in \Sigma$ define

$$\delta_a^Q : D_a \to Q \quad \text{and} \quad \delta_a^{\Gamma^*} : D_a \to \Gamma^*$$

by setting

$$\delta_a(p,\gamma) = (\delta_a^Q(p,\gamma), \delta_a^{\Gamma^*}(p,\gamma)) \in Q \times \Gamma^*$$

for $(p,\gamma) \in D_a$. Put for $k \in \mathbb{Z}_+$

$$D_a(k) = \{(p,\gamma) \in D_a \mid \delta_a^{\Gamma^*}(p,\gamma) \in Q \times \Gamma^k\}.$$

We further assume that a right one-sided subshift Λ_Γ^+ over Γ is given. Let $B_n(\Lambda_\Gamma^+)$ be the set of all admissible words of Λ_Γ^+ of length n. Let us now assume the following conditions:

For $(p, q_0) \in D_a$ and $\gamma_0\gamma_1\cdots\gamma_l \in B_{l+1}(\Lambda_\Gamma^+)$

(i) If $(p,\gamma_0) \in D_a(0)$, then $(\delta_a^Q(p,\gamma_0),\gamma_1) \in D_b$ for some $b \in \Sigma$.

(ii) If $(p,\gamma_0) \in D_a(k)$ for some $k \geq 1$ and $\delta_a^{\Gamma^*}(p,\gamma_0) = \alpha_1\cdots\alpha_k \in B_k(\Lambda_\Gamma^+)$, then $(\delta_a^Q(p,\gamma_0),\alpha_1) \in D_b$ for some $b \in \Sigma$ and $\alpha_1\alpha_2\cdots\alpha_k\gamma_1\cdots\gamma_l \in B_{k+1}(\Lambda_\Gamma^+)$. We set

$$\begin{aligned}
&V_0 = Q,\\
&V_1 = \{(p,\gamma_1) \in Q \times \Gamma \mid (p,\gamma_1) \in D_a \text{ for some } a \in \Sigma\} (= \cup_{a\in\Sigma} D_a),\\
&V_2 = \{(p,\gamma_1\gamma_2) \in Q \times B_2(\Lambda_\Gamma^+) \mid (p,\gamma_1) \in D_a \text{ for some } a \in \Sigma\},\\
&\quad\cdots\\
&V_l = \{(p,\gamma_1\gamma_2\cdots\gamma_l) \in Q \times B_l(\Lambda_\Gamma^+) \mid (p,\gamma_1) \in D_a \text{ for some } a \in \Sigma\},\\
&\quad\cdots
\end{aligned}$$

The map $\iota : V_{l+1} \to V_l$ is defined by deleting the rightmost symbol:

$$\iota(p,\gamma_1\gamma_2\cdots\gamma_l\gamma_{l+1}) = (p,\gamma_1\gamma_2\cdots\gamma_l).$$

For $(p,\gamma_0\gamma_1\cdots\gamma_l) \in V_{l+1}$ and $a \in \Sigma$, suppose that $(p,\gamma_0) \in D_a(k)$ for $k \in \mathbb{Z}_+$ and put $q = \delta_a^Q(p,\gamma_0) \in Q$ and $\alpha_1\cdots\alpha_k = \delta_a^{\Gamma^*}(p,\gamma_0) \in B_k(\Lambda_\Gamma^+)$.

(i) If $k = 0$, define an edge from $(p,\gamma_0\gamma_1\cdots\gamma_l) \in V_{l+1}$ to $(p,\gamma_1\cdots\gamma_l) \in V_l$ labeled a.

(ii) If $1 \leq k \leq l-1$, define an edge from $(p,\gamma_0\gamma_1\cdots\gamma_l) \in V_{l+1}$ to $(q,\alpha_1\cdots\alpha_k\gamma_1\cdots\gamma_{l-k}) \in V_l$ labeled a.

(iii) If $k \geq l$, define an edge from $(p,\gamma_0\gamma_1\cdots\gamma_l) \in V_{l+1}$ to $(q,\alpha_1\cdots\alpha_l) \in V_l$ labeled a.

Assume that the following transitive condition:

For $(q,\mu_1\cdots\mu_l) \in V_l$ there exists an edge from $(p,\gamma_0\gamma_1\cdots\gamma_l) \in V_{l+1}$ to $(q,\mu_1\cdots\mu_l) \in V_l$ labeled a.

We denote by $E_{l+1,l}$ the set of all such edges from V_{l+1} to V_l. We put $E^M = \cup_{l=0}^{\infty} E_{l+1,l}$. We denote by $\lambda^M : E_{l+1,l}^M \to \Sigma$ the labeling map. We set

$$\mathfrak{L}^M = (V, E^M, \lambda, \iota).$$

We denote by $E_{l,l+1}$ the set of edges reversed its arrow of $E_{l+1,l}$. We put $E_M = \cup_{l=0}^{\infty} E_{l,l+1}$. We set

$$\mathfrak{L}_M = (V, E_M, \lambda, \iota).$$

In what follows, we assume that the one sided subshift Λ_Γ^+ is a topological Markov shift.

Proposition 4.

(i) *The system $\mathfrak{L}^M$ becomes a right-resolving upward λ-graph system over Σ and $\mathfrak{L}_M$ becomes a leftt-resolving downward λ-graph system over Σ.*

(ii) *The symbolic dynamics of the language accepted by the pushdown-automaton M coincides with the symbolic dynamics presented by the λ-graph system $\mathfrak{L}^M$. Similarly the symbolic dynamics of the reversed language accepted by the pushdown-automaton M coincides with the symbolic dynamics presented by the λ-graph system $\mathfrak{L}_M$.*

A pushdown-automaton M is said to be *irreducible* if for $(p, \mu_1 \cdots \mu_l), (q, \nu_1 \cdots \nu_l) \in V_l$ there exists $K \in \mathbb{N}$ such that for any $(q, \nu_1 \cdots \nu_l \gamma_{l+1} \cdots \gamma_{l+k}) \in V_{l+k}$ there exist $a_1, \dots, a_K \in \Sigma$ such that

$$(q, \nu_1 \cdots \nu_l \gamma_{l+1} \cdots \gamma_{l+k}) \overset{\delta_{a_1}}{\to} \overset{\delta_{a_2}}{\to} \cdots \overset{\delta_{a_K}}{\to} (p, \mu_1 \cdots \mu_l).$$

A pushdown-automaton M satisfies *condition* (I) if for for $(p, \mu_1 \cdots \mu_l) \in V_l$ there exists $(q, \nu_1 \cdots \nu_{l+N}) \in V_{l+N}$ such that

$$(q, \nu_1 \cdots \gamma_{l+N}) \overset{\delta_{a_1}}{\to} \overset{\delta_{a_2}}{\to} \cdots \overset{\delta_{a_N}}{\to} (p, \mu_1 \cdots \mu_l)$$

and

$$(q, \nu_1 \cdots \gamma_{l+N}) \overset{\delta_{b_1}}{\to} \overset{\delta_{b_2}}{\to} \cdots \overset{\delta_{b_N}}{\to} (p, \mu_1 \cdots \mu_l)$$

for some distinct words $a_1 \cdots a_N$and $b_1 \cdots b_N$. Hence we have that M is irreducible if and only if $\mathfrak{L}_M$ is λ-irreducible, and M satisfies condition (I) if and only if $\mathfrak{L}_M$ satisfies λ-condition (I).

Therefor we have

Proposition 5. *Let M be a pushdown-automaton and $\mathfrak{L}_M$ the associated left-resolving right-λ-graph system. If M is irreducible with condition (I), then the associated C^*-algebra $\mathcal{O}_{\mathfrak{L}_M}$ is simple purely infinite.*

4. Topological Markov Dyck shifts

1. Dyck shifts D_N:

We consider the Dyck shift D_N with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_1, \dots, \alpha_N\}, \Sigma^+ = \{\beta_1, \dots, \beta_N\}$ and the symbols α_i, β_i satisfy (2.1). Put

$$Q = \{p_0\} : \text{ one point}, \qquad \Gamma = \Sigma^+, \qquad \Sigma = \Sigma^- \cup \Sigma^+$$

and

$$\delta_a : D_a \subset Q \times \Gamma \to Q \times \Gamma^* \qquad \text{for } a \in \Sigma$$

is defined by

(i) For $a = \alpha_i \in \Sigma^-$, we set

$$D_{\alpha_i} = \{(p_0, \beta_i)\}, \quad \text{and} \quad \delta_{\alpha_i}(p_0, \beta_i) = (p_0, \emptyset)$$

hence $k = 0$.

(ii) For $a = \beta_i \in \Sigma^+$, we set

$$D_{\beta_i} = \{(p_0, \beta_j) \mid j = 1, \ldots, N\} = \{p_0\} \times \Sigma^+, \quad \text{and} \quad \delta_{\beta_i}(p_0, \beta_j) = (p_0, \beta_i\beta_j)$$

hence $k = 2$.

The subshift Λ_Γ is defined to be

$$\Lambda_\Gamma = \{(\beta_{i_1}, \beta_{i_2}, \ldots) \in (\Sigma^+)^{\mathbb{N}} \mid \beta_{i_1}, \beta_{i_2}, \cdots \in \Sigma^+\} :$$

the right one sided N-full shift. Set the pushdown-automaton M_N by setting

$$M_N = (Q, \Gamma, \Sigma, \delta).$$

Then we have

Theorem 6 ([KM],[Ma3],[Ma5]).

(i) *The λ-graph system $\mathfrak{L}^{M_N}$ defined by M_N is the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ for the Dyck shift D_N, and the presented subshift $\Lambda_{\mathfrak{L}^{M_N}}$ is D_N. That is,*

$$\mathfrak{L}^{M_N} = \mathfrak{L}^{Ch(D_N)}, \qquad \Lambda_{\mathfrak{L}^{M_N}} = D_N.$$

(ii) *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ is unital, separable, nuclear, simple and purely infinite, and it is the unique C^*-algebra generated by N partial isometries $S_i, i = 1, \ldots, N$ and N isometries $T_i, i = 1, \ldots, N$ subject to the relations:*

$$\sum_{j=1}^{N} S_j^* S_j = 1,$$

$$E_{\mu_1\cdots\mu_l} = \sum_{j=1}^{N} S_j S_j^* E_{\mu_1\cdots\mu_l} S_j S_j^* + T_{\mu_1} E_{\mu_2\cdots\mu_l} T_{\mu_1}^*, \quad l = 2, 3, \ldots$$

where $E_{\mu_1\cdots\mu_l} = S_{\mu_1}^* \cdots S_{\mu_l}^* S_{\mu_l} \cdots S_{\mu_1}$ for $\mu_1, \ldots, \mu_l \in \{1, \ldots, N\}$.

(iii) *Its K-groups are*

$$K_0(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{K}, \mathbb{Z}), \qquad K_1(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}) \cong 0$$

where $C(\mathfrak{K}, \mathbb{Z})$ denotes the abelian group of all $\mathbb{Z}$-valued continuous functions on a Cantor discontinuum $\mathfrak{K}$.

(iv) *For a positive real number β, a KMS state on $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ for the gauge action at inverse temperature $\log\beta$ exists if and only if $\beta = N + 1$. The admitted KMS state is unique.*

(v) *Let $\pi_\varphi(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}})''$ be the von Neumann algebra generated by the GNS-representation $\pi_\varphi(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}})$ of the alegbra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ by the unique KMS state φ. Then $\pi_\varphi(\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}})''$ is the injective factor of type $III_{\frac{1}{N+1}}$.*

We note that the value $\log(N+1)$ is the topological entropy of the Dyck shift D_N ([Kr]).

2. Vertex Dyck shifts D_A for an $N \times N$ matrix $A = [A(i,j)]_{i,j=1,\dots,N}$ with entries in $\{0,1\}$:

Assume that each column and row are both not zero vectors. We consider alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_1, \dots, \alpha_N\}, \Sigma^+ = \{\beta_1, \dots, \beta_N\}$ such that the symbols α_i, β_i satisfy (2.1). Put

$$Q = \{p_0\} : \text{ one point}, \qquad \Gamma = \Sigma^+, \qquad \Sigma = \Sigma^- \cup \Sigma^+$$

and

$$\delta_a : D_a \subset Q \times \Gamma \to Q \times \Gamma^* \qquad \text{for } a \in \Sigma$$

is defined by

(i) For $a = \alpha_i \in \Sigma^-$, we set

$$D_{\alpha_i} = \{(p_0, \beta_i)\}, \quad \text{ and } \quad \delta_{\alpha_i}(p_0, \beta_i) = (p_0, \emptyset)$$

hence $k = 0$.

(ii) For $a = \beta_i \in \Sigma^+$, we set

$$D_{\beta_i} = \{(p_0, \beta_j) \mid A(i,j) = 1, j = 1, \dots, N\} \quad \text{ and } \quad \delta_{\beta_i}(p_0, \beta_j) = (p_0, \beta_i\beta_j)$$

hence $k = 2$.

The subshift Λ_Γ is defined to be

$$\Lambda_\Gamma = \{(\beta_{i_1}, \beta_{i_2}, \dots) \in (\Sigma^+)^{\mathbb{N}} \mid A(i_n, i_{n+1}) = 1, n \in \mathbb{N}\} :$$

the right one sided topological Markov shift Λ_A. Set the pushdown-automaton M_A by setting

$$M_A = (Q, \Gamma, \Sigma, \delta).$$

Suppose that A is irreducible with condition (I). Then we have

Theorem 7 ([Ma5]).

(i) *The λ-graph system $\mathfrak{L}^{M_A}$ defined by M_A is the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_A)}$ for the vertex Dyck shift D_A, and the presented subshift $\Lambda_{\mathfrak{L}^{M_A}}$ is D_A. That is,*

$$\mathfrak{L}^{M_A} = \mathfrak{L}^{Ch(D_A)}, \qquad \Lambda_{\mathfrak{L}^{M_A}} = D_A.$$

(ii) *The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_A)}}$ associated with the λ-graph system $\mathfrak{L}^{Ch(D_A)}$ is separable, unital, nuclear, simple and purely infinite, and it is the unique C^*-algebra generated by $2N$ partial isometries $S_i, T_i, i = 1, \dots, N$ subject to the relations:*

$$\sum_{j=1}^{N}(S_jS_j^* + T_jT_j^*) = 1, \qquad \sum_{j=1}^{N} S_j^*S_j = 1,$$

$$T_i^*T_i = \sum_{j=1}^{N} A(i,j)S_j^*S_j, \qquad i = 1, 2, \dots, N,$$

$$E_{\mu_1\cdots\mu_k} = \sum_{j=1}^{N} A(j,\mu_1)S_jS_j^*E_{\mu_1\cdots\mu_k}S_jS_j^* + T_{\mu_1}E_{\mu_2\cdots\mu_k}T_{\mu_1}^*,$$

where $E_{\mu_1\cdots\mu_k} = S^*_{\mu_1}\cdots S^*_{\mu_k}S_{\mu_k}\cdots S_{\mu_1}$, $(\mu_1,\cdots,\mu_k)\in\Lambda^*_A$, *and* Λ^*_A *is the set of admissible words of the topological Markov shift* Λ_A *defined by the matrix* A.

(iii) *For the matrix* $F = \begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$, *the K-groups of the simple, purely infinite* C^**-algebra* $\mathcal{O}_{\mathfrak{L}^{Ch(D_F)}}$ *are*

$$K_0(\mathcal{O}_{\mathfrak{L}^{Ch(D_F)}}) \cong \mathbb{Z}\oplus C(\mathfrak{K},\mathbb{Z})^\infty, \qquad K_1(\mathcal{O}_{\mathfrak{L}^{Ch(D_F)}}) \cong 0$$

where $C(\mathfrak{K},\mathbb{Z})^\infty$ *denotes the countable infinite direct sum of the group* $C(\mathfrak{K},\mathbb{Z})$.

Theorem 6 and Theorem 7 say that the C^*-algebras $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ and $\mathcal{O}_{\mathfrak{L}^{Ch(D_F)}}$ are finitely generated, and its K_0-group however are not finitely generated. Therefore the algebras $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ and $\mathcal{O}_{\mathfrak{L}^{Ch(D_F)}}$ are not semiprojective whereas Cuntz algebras and Cuntz-Krieger algebras are semiprojective

3 Sofic Dyck shift $D_{\mathcal{G}}$ for a labeled graph $\mathcal{G}$:

Let $\mathcal{G} = (V_G, E_G)$ be a finite directed graph whose adjacency matrix A_G satisfies condition (I). Let $\lambda : E_G \to \Sigma^+ = \{\beta_1,\dots,\beta_N\}$ be a bijective map. Then we have a labeled graph $\mathcal{G} = (G,\lambda)$ over Σ^+. Let

$$Q = V_G, \qquad \Gamma = \Sigma^+, \qquad \Sigma = \Sigma^-\cup\Sigma^+$$

and

$$\delta_a : D_a \subset Q\times\Gamma \to Q\times\Gamma^* \qquad \text{for } a\in\Sigma$$

is defined by

(i) For $a = \alpha_i \in \Sigma^-$, we set

$$D_{\alpha_i} = \{(p,\beta_i)\in Q\times\Sigma^+ \mid \text{there exists an edge } e\in E_G; s(e) = p, \lambda(e) = \beta_i\},$$

and

$$\delta_{\alpha_i}(p,\beta_i) = (q,\emptyset) \text{ where } q = t(e) \text{ for } p = s(e), \lambda(e) = \beta_i$$

hence $k = 0$.

(ii) For $a = \beta_i \in \Sigma^+$, we set

$$D_{\beta_i} = \{(p,\beta_j)\in V_G\times E_G \mid \text{there exist } e,f\in E_G; s(e) = t(f) = p, \lambda(e) = \beta_j, \lambda(f) = \beta_i\}$$

and

$$\delta_{\beta_i}(p,\beta_j) = (q,\beta_i\beta_j) \qquad \text{where} \quad q = s(f)$$

hence $k = 2$.

The subshift Λ^+_Γ is defined to be

$$\Lambda^+_\Gamma = \{(\lambda(e_{i_1}),\lambda(e_{i_2}),\dots)\in(\Sigma^+_N)^{\mathbb{N}} \mid \exists e_{i_1},e_{i_2},\dots; t(e_{i_n}) = s(e_{i_{n+1}}), n\in\mathbb{N}\}$$

the right one sided edge shift $\Lambda_{\mathcal{G}}$. Set the pushdown-automaton $M_{\mathcal{G}}$ by setting

$$M_{\mathcal{G}} = (Q,\Gamma,\Sigma,\delta).$$

Then we have

Proposition 8. *The λ-graph system $\mathfrak{L}^{M_{\mathcal{G}}}$ defined by $M_{\mathcal{G}}$ is the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_{\mathcal{G}})}$ for the sofic Dyck shift $D_{\mathcal{G}}$, and the presented subshift $\Lambda_{\mathfrak{L}^{M_{\mathcal{G}}}}$ is the sofic Dyck shift $D_{\mathcal{G}}$. That is,*

$$\mathfrak{L}^{M_{\mathcal{G}}} = \mathfrak{L}^{Ch(D_{\mathcal{G}})}, \qquad \Lambda_{\mathfrak{L}^{M_{\mathcal{G}}}} = D_{\mathcal{G}}.$$

Similar results for $M_{\mathcal{G}}$ to Theorem 7 hold. The pushdown-automaton $M_{\mathcal{G}}$ and the sofic Dyck shift $D_{\mathcal{G}}$ corresponds to the intersection between the Dyck language and the regular language coming from the labeled graph $\mathcal{G}$. We note that any context free language is a homomorphic image of the intersection between a Dyck language and a regular language ([HU]).

The discussions in this section and the preceding section may be genaralized to Turing machines ([Ma5]).

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